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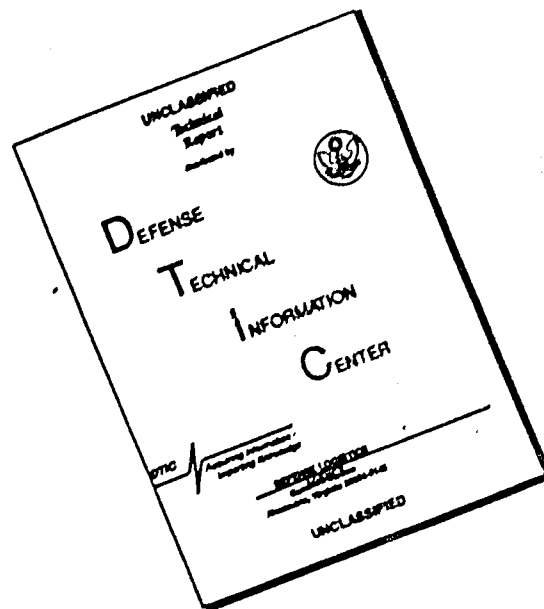
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System Structure and the Existence of a System Life

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SYSTEM STRUCTURE AND THE EXISTENCE

OF A SYSTEM LIFE

by

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0. Summary

The reliability at time t of a system in sustained operation is often taken to be the probability that it functions continuously during the time interval $[0,t]$. The standard computation of system reliability finds the probability that the system functions at time t in terms of the probabilities that its components function at time t. This procedure is relevant only if the system, and its components, have _lives_ (roughly speaking, a device has a life if it functions continuously until some time of failure, and remains failed thereafter).

We show that if each component of a coherent system has a life, then the system has a life (again roughly, a system is coherent if its performance is not impaired by an improvement in the performance of its components). Our principal result is that, under reasonable conditions, the converse is true: if the system has a life, then the system is coherent and each component has a life. This means that if the standard computation of system reliability is to be used, the system in question should be coherent.

1. Introduction, Definitions, Conventions

We consider systems whose performance is determined by the performance of their components. In establishing a binary model of such a system we suppose that each of the n components in the system may be in one of two conditions, functioning or failed. The joint performance of the components is indicated by a vector $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, where $x_i = 1$ or 0 if the ith component is functioning or failed. We suppose that for each possible combination of component performances the system is either functioning or failed. The performance of the system is indicated by its structure function

ϕ, where $\phi x = 1$ or 0 according as the system functions or fails for the joint component performance indicated by x.

A component is _inessential_ to the system if its performance has no effect on the performance of the system, i.e. the ith component is inessential if $\phi(1_i, x) = \phi(0_i, x)$ for all vectors $(\cdot_i, x)$, where

$$(\cdot_i, x) = (x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$$
$$(1_i, x) = (x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$
$$(0_i, x) = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

A component is _essential_ to the system if it is not inessential. A component which is inessential can be deleted from the system.

A system is _coherent_ if its structure function satisfies the conditions

(1.1) $\phi 1 = 1$, where $1 = (1, 1, \ldots, 1)$,

(1.2) $\phi 0 = 0$, where $0 = (0, 0, \ldots, 0)$,

(1.3) $\phi x \geq \phi y$ whenever $x \geq y$ in the sense that $x_i \geq y_i$, $i = 1, 2, \ldots, n$.

There are only two systems which satisfy (1.3) and are not coherent, the system which never functions, $\phi x \equiv 0$, and the system which always functions, $\phi x \equiv 1$. Neither system has any essential components. We will consider only systems that have _at least one_ essential component, _and have no_ inessential components. Such systems are coherent if, and only if, (1.3) is satisfied.

The preceding definitions and notation are presented more fully, and illustrated, in [1] and [2].

Most studies of coherent systems have concentrated on their behavior at some fixed time; we consider here their behavior during a period of time. For this purpose, it is convenient to represent the performance of a device (system or component) by a stochastic process $\{X(t), t \geq 0\}$, where $X(t) = 1$ or 0 according as the device is functioning or failed at time t. In order to avoid certain pathologies, we always make the natural assumption that sample functions $X(t)$ of performance processes are almost surely continuous on the right. When discussing a system in relation to its components we will introduce performance processes $\{X_i(t), t \geq 0\}$, $i = 1, 2, \ldots, n$, for the components and the corresponding performance process $\{\underline{X}(t), t \geq 0\}$ for the system. Right continuity in the sample functions of the component processes is consistent with right continuity in the sample functions of the system process.

2. System and Component Life

Intuitively, a device has a life if it functions continuously in time until failure occurs, after which it remains failed; thus we have

<u>Definition 2.1</u>. A device with performance process $\{X(t), t \geq 0\}$ is said to have a <u>life</u> if

(2.1) $P\{X(s) = 1 \text{ for all } s \text{ in } [0,t) | X(t) = 1\} = 1$

for all t such that $P\{X(t) = 1\} > 0$.

When a device has a life, that life is the duration T of the time interval preceding failure, so that $T > t$ if and only if $X(s) = 1$ for all s in $[0,t]$. It follows, using (2.1) that

$$P\{T > t\} = P\{X(s) = 1 \text{ for all } s \text{ in } [0,t), X(t) = 1\}$$

$$(2.2) \qquad = P\{X(s) = 1 \text{ for all } s \text{ in } [0,t) | X(t) = 1\} \, P\{X(t) = 1\}$$

$$= P\{X(t) = 1\}.$$

Thus when a device has a life, the probability of its functioning throughout the time interval $[0,t]$ is just the probability of its functioning at t. The converse is also true.

Remark.

Most commonly the reliability $r(t)$ of a device at time t is defined as the probability that it functions continuously during $[0,t]$. The standard computation of system reliability $R(t)$ from component reliabilities $r_i(t)$, $i = 1,2,\dots,n$, i.e.

$$R(t) = P\{\phi X(t) = 1\} = \sum_{\{x | \phi x = 1\}} P\{X(t) = x\}$$

$$= \sum_{\{x | \phi x = 1\}} \prod_{i=1}^{n} P\{X_i(t) = x_i\} = \sum_{\{x | \phi x = 1\}} \prod_{i=1}^{n} r_i(t)^{x_i}(1 - r_i(t))^{1-x_i},$$

requires the assumptions that component performance processes are independent and that the system and the components all have lives.

Since the performance process is right continuous, it can be readily shown that (2.1) is equivalent to

$$(2.3) \qquad P\{X(s) = 1 | X(t) = 1\} = 1$$

whenever $P\{X(t) = 1\} > 0$ and $0 \leq s < t$. This, of course, is equivalent to

$$(2.4) \qquad P\{X(s) = 0, X(t) = 1\} = 0$$

whenever $0 \leq s < t$.

Except where noted we do not assume component performance processes to be independent. The following theorem gives a useful condition equivalent to the existence of component lives.

Theorem 2.1. Each component in a system has a life if and only if

(2.5) $P\{\underline{X}(s) = \underline{x}, \underline{X}(t) = \underline{y}\} = 0$

whenever $\underline{x} \not\geq \underline{y}$ and $0 \leq s < t$.

Proof.

Suppose first that each component has a life. Let $\underline{x} \not\geq \underline{y}$ ($\underline{x} \geq \underline{y}$ unless $x_i = 0$ and $y_i = 1$ for some i), and choose i for which $x_i = 0$, $y_i = 1$. Then

$$P\{\underline{X}(s) = \underline{x}, \underline{X}(t) = \underline{y}\} \leq P\{X_i(s) = 0, X_i(t) = 1\} = 0$$

so that (2.5) is satisfied.

Next suppose that the ith component does not have a life. Then for some $s < t$,

$$0 < P\{X_i(s) = 0, X_i(t) = 1\} = \sum_{(0_i, \underline{x})} \sum_{(1_i, \underline{y})} P\{\underline{X}(s) = (0_i, \underline{x}), \underline{X}(t) = (1_i, \underline{y})\}.$$

Thus for at least one choice of $(0_i, \underline{x})$ and $(1_i, \underline{y})$,

$$P\{\underline{X}(s) = (0_i, \underline{x}), \underline{X}(t) = (1_i, \underline{y})\} > 0,$$

a contradiction of (2.5). ‖

It is easily seen that (2.5) is equivalent to

(2.6) $P\{\underline{X}(s) \geq \underline{y} | \underline{X}(t) = \underline{y}\} = 1$

for every $\underline{y}$ and t such that $P\{\underline{X}(t) = \underline{y}\} > 0$, and all $0 \leq s < t$.

3. Relationships between Coherence and the Existence of System and Component Lives

Theorem 3.1. If each component of a coherent system has a life, then the system has a life.

Proof.

From (2.3) it is sufficient to show that $P\{\phi X(s) = 1, \phi X(t) = 1\} = P\{\phi X(t) = 1\}$ whenever $0 \leq s < t$. Observe that

$$P\{\phi X(s) = 1, \phi X(t) = 1\} = \sum_A P\{X(s) = x, X(t) = y\},$$

$$P\{\phi X(t) = 1\} = \sum_B P\{X(s) = x, X(t) = y\}$$

where $A = \{(x,y) \mid \phi x = 1, \phi y = 1\}$, $B = \{(x,y) \mid \phi y = 1\}$. From (2.5) the summand in the expressions above is zero outside $C = \{(x,y) \mid x \geq y\}$. Since the system is coherent $A \cap C = B \cap C$. ‖

We prove two converse forms of Theorem 3.1, both of which require some restrictions (tending in the direction of the usual reliability assumptions) on the joint performance process of the components. The condition (*) of the next theorem is weaker than asserting that components may fail in any order (in time) with positive probability.

Theorem 3.2. If the joint component performance process $\{X(t), t \geq 0\}$ satisfies the condition that

(*) for every $(\cdot_i, x)$ there exists some s and t, $0 \leq s < t$, such that

$$P\{X(s) = (1_i, x), X(t) = (0_i, x)\} > 0,$$

and if the system has a life, then the system is coherent.

Proof.

Suppose that the system is not coherent, so that there is a $(\cdot_i, \underline{x})$ satisfying $\Phi(1_i, \underline{x}) = 0$ and $\Phi(0_i, \underline{x}) = 1$. By $(\cdot)$, there exists s and t, $0 \leq s < t$ such that $0 < P\{\underline{X}(s) = (1_i, \underline{x}), \underline{X}(t) = (0_i, \underline{x})\} \leq P\{\Phi\underline{X}(s) = 0, \Phi\underline{X}(t) = 1\}$, contradicting the condition (2.4) that the system have a life. ‖

A somewhat stronger converse of Theorem 3.1 is obtainable when $(\cdot)$ is replaced by the stronger condition $(\cdot\cdot)$ below.

Theorem 3.3. If the conditions

$(\cdot\cdot)$

    (i)   the component performance processes are independent, and

    (ii) for all $0 \leq s < t$ and $x \geq y$, $P\{X_i(s) = x, X_i(t) = y\} > 0$,

        $i = 1, 2, \ldots, n$,

are satisfied, and the system has a life, then the system is coherent and each component has a life.

Proof.

From $(\cdot\cdot)$, it follows that for every $(\cdot_i, \underline{x})$ and $0 \leq s < t$,

$$P\{\underline{X}(s) = (1_i, \underline{x}), \underline{X}(t) = (0_i, \underline{x})\} = P\{X_i(s) = 1, X_i(t) = 0\} \prod_{j \neq i} P\{X_j(s) = x_j, X_j(t) = x_j\} > 0,$$

which is $(\cdot)$. Thus $(\cdot\cdot)$ is indeed a strengthing of $(\cdot)$, so Theorem 3.2 shows the system to be coherent.

Suppose that the ith component (assumed essential in Section 2) does not have a life. Then there exists a $(\cdot_i, \underline{x})$ such that $\Phi(1_i, \underline{x}) = 1$, $\Phi(0_i, \underline{x}) = 0$, and there exist s and t, $0 \leq s < t$ such that $P\{X_i(s) = 0, X_i(t) = 1\} > 0$. Hence

$$P\{\psi\underline{X}(s) = 0, \phi\underline{x}(t) = 1\} \geq P\{\underline{X}(s) = (0_i, \underline{x}), \underline{X}(t) = (1_i, \underline{x})\}$$

$$= P\{X_i(s) = 0, X_i(t) = 1\} \prod_{j \neq i} P\{X_j(s) = x_j, X_j(t) = x_j\} > 0,$$

a contradiction of (2.4). $\|$

References

[1] Z. W. Birnbaum, J. D. Esary, and S. C. Saunders, "Multi-Component Systems and Structures and Their Reliability," Technometrics, Vol. 3 (1961), pp. 55-77.

[2] J. D. Esary and F. Proschan, "Coherent Structures of Non-Identical Components," Technometrics, Vol. 5 (1963), pp. 191-209.